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THE SIZE FUNCTION ON ABELIAN VARIETIES

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Abstract. The size function is defined for points in projective space over any field K, finitely generated field over Q, generalizing the height function for number fields. We prove that the size function on the K-rational points of an abelian variety is bounded by a quadratic function.

Introduction. In his book, *Introduction to transcendental numbers*, Lang showed how one can extend some of the theorems about the exponential function e^x to theorems about the exponential map from complex g-space to the complex points of group varieties of dimension g, defined over the complex numbers. Looking at transcendental numbers in this general setting, he raised an arithmetic-geometric question about the addition formula of a group variety. In this paper, we shall answer this question in the case of an abelian variety.

In his report to *Seminaire Bourbaki* in May 1964, [6], Lang described the following result of Neron and Tate: If A is an abelian variety defined over a number field K, there exist a quadratic function Q and a linear function L from $A(K)$, the K-rational points of A, to the real numbers, such that the logarithmic height function, $h: A(K) \to R$, defined with respect to any closed immersion in projective space, is additively equivalent to the function $Q+L$. Our main result, Theorem 3.5, is a generalization of this (albeit in a weaker form), to the size function, which is defined for an abelian variety defined over any field of characteristic 0. It states that there is a quadratic function $Q: A(K) \to R$ such that $\text{size}(x) \leqq Q(x)$ for all $x \in A(K)$.

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1. Let K be a field which is finitely generated over Q. K has a *proper set of generators* $\{t_1, \ldots, t_r, u\}$ over Q, denoted $\{t, u\}$, where proper means that $\{t_1, \ldots, t_r\}$ is a transcendence base of K over Q and u is integral over $Z[t_1, \ldots, t_r]$. Let $q = [K: Q(t)]$. An element $\alpha \in K$ is said to be an *integral coordinate* with respect to $\{t, u\}$ if, when α is expressed as a linear combination of $\{1, u, \ldots, u^{q-1}\}$ with coefficients in $Q(t)$ in lowest terms, all coefficients lie in $Z[t]$. Note that if $\alpha, \beta \in K$ are integral coordinates with respect to $\{t, u\}$, then the sum $\alpha+\beta$ and the product $\alpha\beta$ are integral coordinates with respect to $\{t, u\}$.

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Let $\alpha \in K$ be an integral coordinate with respect to $\{t, u\}$, say $\alpha = f_0(t) + \cdots + f_{q-1}(t)u^{q-1}$. We define the *measure* of α with respect to $\{t, u\}$, denoted $|\alpha|_{\{t,u\}}$, to be the maximum of the absolute values of the coefficients of the polynomials $\{f_0, \ldots, f_{q-1}\}$. We define the *degree* of α with respect to $\{t, u\}$, denoted $\deg_{\{t,u\}}(\alpha)$, to be the maximum of the degrees of the polynomials $\{f_0, \ldots, f_{q-1}\}$. Finally, we define the *size* of α with respect to $\{t, u\}$, denoted $\operatorname{size}_{\{t,u\}}(\alpha)$, to be the maximum of the degree of α and the logarithm of the measure of α.

PROPOSITION 1.1. *Let $\alpha_1, \ldots, \alpha_s$ be integral coordinates of K with respect to $\{t, u\}$. Then*

(1) $$\deg_{\{t,u\}}(\alpha_1 + \cdots + \alpha_s) \leqq \max\{\deg_{\{t,u\}}(\alpha_i)\}.$$

(2) $$|\alpha_1 + \cdots + \alpha_s|_{\{t,u\}} \leqq |\alpha_1|_{\{t,u\}} + \cdots + |\alpha_s|_{\{t,u\}}.$$

(3) $$\operatorname{size}_{\{t,u\}}(\alpha_1 + \cdots + \alpha_s) \leqq \operatorname{size}_{\{t,u\}}(\alpha_1) + \cdots + \operatorname{size}_{\{t,u\}}(\alpha_s).$$

(4) $$\deg_{\{t,u\}}(\alpha_1 \cdots \alpha_s) \leqq \deg_{\{t,u\}}(\alpha_1) + \cdots + \deg_{\{t,u\}}(\alpha_s) + \mathfrak{A}(s-1)$$

where $\mathfrak{A}$ depends only on the set $\{t, u\}$.

(5) $$|\alpha_1 \cdots \alpha_s|_{\{t,u\}} \leqq |\alpha_1|_{\{t,u\}} \cdots |\alpha_s|_{\{t,u\}} \prod (\deg_{\{t,u\}}(\alpha_i))^{r+1} \mathfrak{B}^{s-1}$$

where $\mathfrak{B}$ depends only on the set $\{t, u\}$.

(6) $$\operatorname{size}_{\{t,u\}}(\alpha_1 \cdots \alpha_s) \leqq \mathfrak{C}(\operatorname{size}_{\{t,u\}}(\alpha_1) + \cdots + \operatorname{size}_{\{t,u\}}(\alpha_s))$$

where $\mathfrak{C}$ depends only on the set $\{t, u\}$.

Proof. Straightforward, cf. [5, p. 49].

Let $P = P_K^n$ be projective n-space over K, and $x \in P(K)$ a K-rational point of P. If $x = (\alpha_0, \ldots, \alpha_n)$ where all the $\alpha_i \in K$ are integral coordinates with respect to $\{t, u\}$, we say $(\alpha_0, \ldots, \alpha_n)$ are integral coordinates of x with respect to $\{t, u\}$. The *measure* (resp. *degree*, *size*) of x with respect to $\{t, u\}$ denoted $|x|_{\{t,u\}}$ (resp. $\deg_{\{t,u\}}(x)$, $\operatorname{size}_{\{t,u\}}(x)$), is defined to be the greatest lower bound, over all integral coordinates $(\alpha_0, \ldots, \alpha_n)$ of x, of the numbers $\max\{|\alpha_i|_{\{t,u\}}\}$ (resp. $\max\{\deg_{\{t,u\}}(\alpha_i)\}$, $\max\{\operatorname{size}_{\{t,u\}}(\alpha_i)\}$).

Let S be a set and $f_1, f_2 \colon S \to R$ functions taking S into the real numbers. The functions f_1 and f_2 are said to be *additively equivalent* if there exist numbers C_1 and C_2 such that $f_1(s) + C_1 \leqq f_2(s) \leqq f_1(s) + C_2$ for all $s \in S$. The functions f_1 and f_2 are said to be (*multiplicatively*) *equivalent*, denoted $f_1 \sim f_2$, if there exist numbers $C_1, C_2 > 0$ such that $C_1 f_1(s) \leqq f_2(s) \leqq C_2 f_1(s)$ for all $s \in S$. Clearly if f_1 and f_2 are $\geqq \frac{1}{2}$ and additively equivalent, they are also equivalent. We shall be concerned mostly with equivalence classes of functions.

PROPOSITION 1.2. *Let $\{t, u\}$ and $\{t', u'\}$ be two proper sets of generators of K over Q. Then $\operatorname{size}_{\{t,u\}} \sim \operatorname{size}_{\{t',u'\}}$ as functions from $P(K)$ to R.*

Proof. Straightforward.

Proposition 1.2 shows that, up to equivalence, the size function is independent of the choice of a proper set of generators of K over Q. From now on, we shall omit the subscripts referring to the proper set of generators and assume we have picked once and for all a proper set of generators $\{t, u\}$.

LEMMA 1.3. *Let $f_1, \ldots, f_p$ be polynomials in $n+1$ variables with coefficients in K. There exist numbers $F', F''>0$ such that if $\{\alpha_0, \ldots, \alpha_n\}$ is a set of integral coordinates of K and $y=(f_0(\alpha), \ldots, f_p(\alpha))$ is a point of $\boldsymbol{P}_K^{p-1}(K)$, then* $\operatorname{size}(y) \leqq F' \max\{\operatorname{size}(\alpha_i)\} + F''$.

Proof. Straightforward.

Let k be a field, $((\mathrm{Sch}/k))$ the category of k-schemes, $P=\boldsymbol{P}_k^n$, projective n-space over k. For each k-scheme X, consider the set of $(n+2)$-tuples $(L; s_0, \ldots, s_n)$ where L is an invertible sheaf on X and $(s_0, \ldots, s_n)$ is a set of global sections of L which generate L. We say $(L; s_0, \ldots, s_n)$ is isomorphic to $(L'; s_0', \ldots, s_n')$ if there is an isomorphism $v: L \xrightarrow{\sim} L'$ such that $v(s_i)=s_i'$, for $0 \leqq i \leqq n$. If we let $M(X)$ be the set of isomorphism classes, it is clear that M is a contravariant functor of X.

For any k-morphism $f: X \rightarrow P$, the element $(\mathcal{O}_P(1); H_0, \ldots, H_n) \in M(P)$, where the H_i are generating hyperplanes of P, yields $(f^*\mathcal{O}_P(1); f^*H_0, \ldots, f^*H_n)$ in $M(X)$. Thus we have, for each k-scheme X, a natural map $T_X: P(X) \rightarrow M(X)$. In fact, the collection of the T_X is an isomorphism of functors [7, p. 31], so in particular each T_X is a bijection.

Let $f: X \rightarrow P$ be a k-morphism, $x \in X(k)$. Then $f(x)$ is a k-rational point of P. Let $(a_0, \ldots, a_n)$ be any set of coordinates of $f(x)$ with $a_i \in k$. If $(L; s_0, \ldots, s_n) = T_X(f)$, then $s_i(x)=aa_i$ for some $a \in k^*$, $0 \leqq i \leqq n$, depending on the identification $u: L_x/m_xL_x \xrightarrow{\sim} k(x)$. Furthermore, for any two such identifications u and u', we have $u(s_i(x))=bu'(s_i(x))$, $0 \leqq i \leqq n$, for some unit $b \in k^*$. Note that if $(L; s_0, \ldots, s_n)$ and $(L'; s_0', \ldots, s_n')$ are isomorphic and $x \in X(k)$, then $(s_0(x), \ldots, s_n(x)) = (s_0'(x), \ldots, s_n'(x)) \in P(k)$.

Now let K be a field which is finitely generated over Q, and let $P=\boldsymbol{P}_K^n$.

DEFINITION. Let X be a K-scheme, $f: X \rightarrow P$ a K-morphism. The *size function* with respect to f, denoted $\operatorname{size}_f: X(K) \rightarrow R$, is the function $\operatorname{size}_f(x)=\operatorname{size}(f(x))$ for all $x \in X(K)$. Note that this depends only on the isomorphism class of $T_X(f)$.

PROPOSITION 1.4. *Let X, Y be K-schemes, $g: Y \rightarrow X, f: X \rightarrow P$, K-morphisms. Then the functions $\operatorname{size}_{f\circ g}$ and $(\operatorname{size}_f)\circ g$ from $Y(K)$ to R are equal.*

Proof. Let $(L; s_0, \ldots, s_n)=T_X(f)$, and let $y \in Y(K)$. Then $\operatorname{size}_f(g(y))$ is the size of $(s_0(g(y)), \ldots, s_n(g(y))) \in P(K)$, while $\operatorname{size}_{f\circ g}(y)$ is the size of $((g^*s_0)(y), \ldots, (g^*s_n)(y)) \in P(K)$. But $g^*s_i=g^*f^*(H_i)=(f\circ g)^*(H_i)$; so, $g^*s_i(y)=s_i(g(y))$.

PROPOSITION 1.5. *Let X be a quasi-projective K-scheme (with a fixed embedding in P), $f: X \rightarrow \boldsymbol{P}_K^m$ a K-morphism. There exists a number F_1 depending only on f such that* $\operatorname{size}_f(x) \leqq F_1 \operatorname{size}(x)$ *for all $x \in X(K)$.*

Proof. Given $\varepsilon > 0$, let $(\alpha_0, \ldots, \alpha_n)$ be integral coordinates of $x \in X(K)$ such that $\text{size}(\alpha_i) \leqq \text{size}(x) + \varepsilon$. The coordinates β_i of $f(x)$ are polynomials in the α_i with coefficients in K, so, by Lemma 1.3, there exists a number F_1 independent of ε and x such that $\text{size}_f(x) \leqq F_1(\text{size}(x) + \varepsilon)$. Since ε was arbitrary, the conclusion follows.

Suppose X is a proper K-scheme; let L be an invertible sheaf on X. By the finiteness theorem [2, III 3.2.1], $H^0(X, L)$ is a finitely generated K-module. If the global sections of L generate L, then a basis of $H^0(X, L)$ defines a K-morphism $f\colon X \to \boldsymbol{P}_K^n$ for some positive integer n. Therefore if L is an invertible sheaf on X whose global sections generate it, we may define size_L to be the function size_f where $T_X(f) = (L; s_0, \ldots, s_n)$ and $\{s_0, \ldots, s_n\}$ is a basis of $H^0(X, L)$. If $\{s'_0, \ldots, s'_m\}$ is any set of generators of $H^0(X, L)$ and $T_X(g) = (L; s'_0, \ldots, s'_m)$, then $\text{size}_L \sim \text{size}_g$ by Proposition 1.5.

2. Let G be a commutative group scheme defined over K where K is a field, finitely generated over Q. Let $P = \boldsymbol{P}_K^n$ be projective n-space K, and let $\{t_1, \ldots, t_r, u\}$ be a proper set of generators of K over Q. For each integer N, we have an endomorphism $(N) = N_G$ of G which takes each $x \in G(K)$ into $Nx \in G(K)$.

PROPOSITION 2.1. *Let $f\colon G \to P$ be a K-morphism, and suppose f has the property that, for some given positive integer N, there exist $n+1$ homogeneous polynomials $\sum c^i_{(k)} Z_{(k)} \in K[Z]$, $0 \leqq i \leqq n$, of degree N^2 such that, for $x \in G(K)$, the ith coordinate of $f(Nx)$ can be written $\sum c^i_{(k)} s_{(k)}(x)$, where $T_G(f) = (L; s_0, \ldots, s_n)$. (Here $Z_{(k)}$ denotes $Z_{k_1} \cdots Z_{k_{N^2}}$.) Then there exists a number C' (depending on N) independent of x and m such that*

$$\text{size}_f(N^m x) \leqq N^{2m} R\, \text{size}_f(x) + C' N^{2m}$$

where $R = 1 + N^2(r+1)/(N^2 - 1)$ for all integers $m \geqq 0$.

Proof. Throughout the proof, $\deg(x)$ (resp. $|x|$) will refer to $\deg(f(x))$ (resp. $|f(x)|$). We may assume that each coefficient $c^i_{(k)}$ is an integral coordinate and that $s_i(Nx) = \sum c^i_{(k)} s_{(k)}(x)$. Furthermore, for any $\varepsilon > 0$, we may assume that the $s_i(x)$ are integral coordinates and that $\text{size}(s_i(x)) \leqq \text{size}_f(x) + \varepsilon$; hence we may assume $\text{size}(s_i(x)) \leqq \text{size}_f(x)$, $0 \leqq i \leqq n$, for all $x \in G(K)$.

Let $C_1 = \max\{\deg(c^i_{(k)})\} + N^2 \mathfrak{A}$. Then one shows by an easy induction on m that $\deg(N^m x) \leqq N^{2m} \deg(x) + D_m C_1$ where $D_m = (N^{2m} - 1)/(N^2 - 1)$. Thus we have

$$\deg(N^m x) \leqq N^{2m}(\deg(x) + C_1). \tag{7}$$

In particular, there is a number C_2, independent of x and m such that $\deg(N^m x) \leqq C_2 N^{2m} \deg(x)$. Now let $C_3 = \max\{|c^i_{(k)}|\} C_1^{r+1}((n+1)\mathfrak{B} C_2^{r+1})^{N^2}$. Then by a similar induction on m, one shows that

$$|N^m x| \leqq (C_3 (\deg(x))^{N^2(r+1)})^{D_m} N^{E_m} |x|^{N^{2m}} \tag{8}$$

where $E_m = N^2(r+1)(N^{2m} - mN^2 + m - 1)/(N^2 - 1)^2$. Combining inequalities (7) and (8), one obtains Proposition 2.1.

By Lemma 1.3, the addition map $s_{12}\colon G\times G\to G$ induces an inequality

$$\text{size}_f(x+y) \leqq T(\text{size}_f(x)+\text{size}_f(y)) \tag{9}$$

for all $x, y\in G(K)$ where T is independent of x and y. Let p be an integer such that $T\leqq N^p$.

PROPOSITION 2.2. *Under the hypotheses of Proposition* 2.1, *there exists a number D, independent of $m\in Z$ and $x\in G(K)$, such that*

$$\text{size}_f(mx) \leqq Dm^2\,\text{size}_f(x). \tag{10}$$

Proof. It follows from (9) that $\text{size}_f(m'x)\leqq\{T(T^{m'}-1)/(T-1)\}\,\text{size}_f(x)$. Let $\sigma(x)=\max\{\text{size}_f(m'x)\}$, $0\leqq m'<N^{p+1}$. Then

$$\sigma(x) \leqq \{T(T^{N^{p+1}}-1)/(T-1)\}\,\text{size}_f(x).$$

By Proposition 1.5, there exists a number C_4 such that $\text{size}_f(-x)\leqq C_4\,\text{size}_f(x)$. Therefore to prove Proposition 2.2, it suffices to show that

$$\text{size}_f(mx) \leqq (RN^2+1)m^2\sigma(x)+m^2C'N^2. \tag{11}$$

LEMMA 2.3. *If $m<N^{s+1}$, then*

$$\text{size}_f(mx) \leqq TN^{2(s-p)}(R\sigma(x)+C')+T\,\text{size}_f(m_1x), \tag{12}$$

where $m_1<N^{s-p}$.

Proof. For some $s'\leqq s$, we have $N^{s'}\leqq m<N^{s'+1}$. Divide the interval, $[N^{s'}, N^{s'+1}]$, into $N^{p+1}-N^p$ equal intervals, $[N^{s'}+MN^{s'-p}, N^{s'}+(M+1)N^{s'-p}]$ for $0\leqq M<N^{p+1}-N^p$. Then for some M, we have $N^{s'}+MN^{s'-p}\leqq m<N^{s'}+(M+1)N^{s'-p}$. Therefore we have

$$\text{size}_f(mx) \leqq TN^{2(s'-p)}(R\,\text{size}_f((N^p+M)x)+C')+T\,\text{size}_f(\{(m-(N^p+M))N^{s'-p}\}x).$$

This proves Lemma 2.3.

LEMMA 2.4. *If i is a nonnegative integer, then*

$$\text{size}_f(mx) \leqq (R\sigma(x)+C')N^{2(s+1)}+T^i\,\text{size}_f(m_ix)$$

where either $m_i<N^{s-i(p+1)+1}$ or $m_i<N^{p+1}$.

Proof. One first proves by induction on i, using Lemma 2.3, that

$$\text{size}_f(mx) \leqq (R\sigma(x)+C')\{TN^{2(s-p)}+\cdots+T^iN^{2(s-i(p+1)+1)}\}+T^i\,\text{size}_f(m_ix)$$

where $m_i<N^{s-i(p+1)+1}$ or $m_i<N^{p+1}$. Then since $\sum_{j=1}^{i}T^jN^{2(s-j(p+1)+1)}\leqq N^{2(s+1)}/(N^2-1)$ the proof of Lemma 2.4 is complete.

Let s be the integer such that $N^s\leqq m<N^{s+1}$. If $s\leqq p$, then (11) is trivial. Hence we may assume $s>p$. Let i be the integer such that $s/p\geqq i>(s/p)-1$. Then $m_i<N^{p+1}$ and hence $\text{size}_f(m_ix)\leqq\sigma(x)$. Furthermore $T^i\leqq N^s$. Therefore, by Lemma 2.4, we

have $\text{size}_f(mx) \leqq (R\sigma(x)+C')N^{2(s+1)}+N^s\sigma(x) \leqq (RN^2+1)m^2\sigma(x)+m^2C'N^2$. The proof of Proposition 2.2 is now complete.

3. Let k be a field, $P=\boldsymbol{P}_k^n$.

PROPOSITION 3.1. *Let X be a projective scheme over k, $f\colon X \hookrightarrow P$ a k-immersion and let $(L; s_0, \ldots, s_n)=T_X(f)$. Then there exists a positive integer N_0 such that, for all integers $N' \geqq N_0$, the k-module of global sections of $L^{\otimes N'}$ is generated by monomials of degree N' of the global sections $\{s_i\}$ of L.*

Proof. For each integer N', we have an exact sequence of coherent $\mathcal{O}_P$-modules

$$0 \to I(N') \to \mathcal{O}_P(N') \to L^{\otimes N'} \to 0$$

where I is the sheaf of ideals defining the closed immersion $f\colon X \hookrightarrow P$. By Serre's theorem [2, III 2.2.2(iii)], there exists an integer N_0 such that $H^1(P, I(N'))=0$ for all $N' \geqq N_0$. Furthermore, $H^0(P, \mathcal{O}_P(N'))$ is equal to the k-module generated by the monomials of degree N' of global sections of $\mathcal{O}_P(1)$ [2, III 2.1.12(ii)]. Hence for $N' \geqq N_0$, the map

$$H^0(P, \mathcal{O}_P(1))^{\otimes N'} \to H^0(X, L^{\otimes N'})$$

is surjective; Proposition 3.1 now follows easily.

Let W_1, W_2 be algebraic k-schemes. Let $p_i\colon W_1 \times W_2 \to W_i$, $i=1, 2$, be the projection on the ith factor. If w is a k-rational point of W_1, then let p_w be the composition

$$W_1 \longrightarrow \text{Spec}\,(k) \xrightarrow{\sim} \text{Spec}\,(k(w)) \xrightarrow{\{w\}} W_1$$

where $\{w\}\colon \text{Spec}(k(w)) \to W_1$ is the canonical closed immersion.

PROPOSITION 3.2. *Let X, Y, Z be proper varieties defined over k; let $x_0 \in X$, $y_0 \in Y$, and $z_0 \in Z$ be k-rational points and let M be an invertible sheaf on $X \times Y \times Z$. Suppose $M|X \times Y \times \{z_0\}$ (i.e., $(\text{id}_X \times \text{id}_Y \times \{z_0\})^*M$), $M|X \times \{y_0\} \times Z$, and $M|\{x_0\} \times Y \times Z$ are trivial. Then M itself is trivial.*

Proof. Let $d=(\text{id}_X, p_{x_0}) \times (\text{id}_Y, p_{y_0}) \times (\text{id}_Z, p_{z_0})$, $d\colon X \times Y \times Z \to X \times X \times Y \times Y \times Z \times Z$, and let $p_{ijk}=p_i \times p_j \times p_k$, $p_{ijk}\colon X \times X \times Y \times Y \times Z \times Z \to X \times Y \times Z$ for $i, j, k =1, 2$. Then $M'=\sum p_{ijk}{}^*M^{\otimes(i+j+k-3)} \simeq 0$ by the theorem of the cube [3, p. 68], so $d^*M' \simeq \mathcal{O}_{X\times Y\times Z}$. On the other hand, $d^*p_{111}{}^*M \simeq M$ and $d^*p_{ijk}{}^*M \simeq \mathcal{O}_{X\times Y\times Z}$ if $ijk \neq 1$ by assumption. Therefore $M \simeq \mathcal{O}_{X\times Y\times Z}$.

COROLLARY 3.3. *Let $f, g, h\colon S \to A$ be k-morphisms from a k-scheme S to an abelian variety A, and let L be an invertible sheaf on A. Then*

$$(13) \qquad (fgh)^*L \simeq (fg)^*L \otimes (fh)^*L \otimes (gh)^*L \otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1},$$

where fg means the product of f and g.

Proof. Let e be the unit element of A and $s_{123}\colon A\times A\times A\to A$ be the group addition. Let $s_{12}=s_{123}\circ(\mathrm{id}_A\times\mathrm{id}_A\times p_e)$, $s_{13}=s_{123}\circ(\mathrm{id}_A\times p_e\times\mathrm{id}_A)$ and $s_{23}=s_{123}\circ(p_e\times\mathrm{id}_A\times\mathrm{id}_A)$. Then applying Proposition 3.2 to $X=Y=Z=A$, $x_0=y_0=z_0=e$ and $M=s_{123}{}^*L\otimes s_{12}{}^*L^{-1}\otimes s_{13}{}^*L^{-1}\otimes s_{23}{}^*L^{-1}\otimes p_1{}^*L\otimes p_2{}^*L\otimes p_3{}^*L$ we obtain $s_{123}{}^*L\cong s_{12}{}^*L\otimes s_{13}{}^*L\otimes s_{23}{}^*L\otimes p_1{}^*L^{-1}\otimes p_2{}^*L^{-1}\otimes p_3{}^*L^{-1}$. Pulling this isomorphism back to S via $(f,g,h)\colon S\to A\times A\times A$, we obtain (13).

COROLLARY 3.4. *Let A be an abelian variety over k, L an invertible sheaf on A, and let $L'=(-1)^*L$ then*

$$(N)^*L \cong L^{\otimes(N^2+N)/2}\otimes L'^{\otimes(N^2-N)/2} \tag{14}$$

for all integers $N>0$. In particular, if L is symmetric, (i.e., $L=L'$), then $(N)^*L\cong L^{\otimes N^2}$.

Proof. The proof proceeds by induction on N. The conclusion is trivial for $N=1$. Assume (14) holds for all positive integers $<N+1$. By (13) with $f=(N)$, $g=(1)$, and $h=(-1)$, $(N+1)^*L\cong(N)^*L^2\otimes(N-1)^*L^{-1}\otimes L\otimes L'$. Hence by induction we have the result.

Let K be a field, finitely generated over Q. Let A be an abelian variety defined over K. Since A is projective [3, p. 87], there exists a very ample symmetric invertible sheaf L on A. Indeed, if L_1 is an invertible sheaf on A, then $L_1\otimes L_1'$ is symmetric; furthermore, if L_1 is very ample, so is $L_1\otimes L_1'$ by [2, II 4.4.9].

THEOREM 3.5. *Let A be an abelian variety defined over K. If L is a very ample invertible sheaf on A, then the size function* $\mathrm{size}_L\colon A(K)\to R$ *is bounded by a quadratic function, i.e. there exists a quadratic function $Q\colon A(K)\to R$ such that* $\mathrm{size}_L(x)\leqq Q(x)$ *for all $x\in A(K)$.*

Proof. We may assume L is symmetric. Indeed, any two closed immersions of A differ from one another by an isomorphism, so their corresponding size functions are equivalent by Proposition 1.5. Let f be a corresponding immersion.

By Corollary 3.4, for each $N>0$ we have $(N)^*L\cong L^{\otimes N^2}$. If $N^2\geqq N_0$, then by Proposition 3.1 each of the global sections $s_i\circ(N)\in L^{\otimes N^2}$ (where $\{s_i\}$ is a basis of $H^0(A,L)$) can be expressed as a polynomial of degree N^2 of the sections s_i with coefficients in K, i.e. $s_i(Nx)=\sum c^i_{(k)}s_{(k)}(x)$ where $c^i_{(k)}\in K$ and $(k)\in Z^{N^2}$. Therefore, $f\colon A\to P^n_K$ satisfies the conditions of Proposition 2.1.

By the Mordell-Weil Theorem [4, p. 71], $A(K)$ is a finitely generated group. Choose a direct sum decomposition of $A(K)$ into a free subgroup and a torsion subgroup. Since both direct summands are finitely generated, we may choose a basis $\{x_1,\dots,x_l\}$ of the free summand and a constant C such that $\mathrm{size}_L(x_t)\leqq C$ for all x_t in the torsion summand.

Each element $x\in A(K)$ can be uniquely expressed in the form $x=n_1x_1+\cdots+n_lx_l+x_t$ where $n_i\in Z$ and x_t is a torsion element of $A(K)$. Therefore by Lemma 1.3 there exists a number T such that

$$\mathrm{size}_L(x)\leqq T(\mathrm{size}_L(n_1x_1)+\cdots+\mathrm{size}_L(n_lx_l)+C).$$

By Proposition 2.2, we have $\operatorname{size}_L(x) \leqq Q(x)$ where

$$Q(x) = TD(n_1^2 \operatorname{size}_L(x_1) + \cdots + n_l^2 \operatorname{size}_L(x_l) + C)$$

and Q is clearly quadratic.

Let G be an affine group variety. Then one can use the results of §2 to prove a result similar to Theorem 3.5 for affine group varieties. However, a direct proof is easier.

PROPOSITION 3.6. *Let G be an affine group variety defined over K. Then there exists a number C' such that*

$$\operatorname{size}(x^N) \leqq C'|N| \operatorname{size}(x)$$

for all $N \in Z$ and all $x \in G(K)$.

Proof. By Proposition 1.5, we may assume $N > 0$. The group variety G is contained in $\mathrm{GL}(M, K)$ for some integer $M > 0$ [1, p. 4–03]. Let $x = \|a_{ij}\|$, $1 \leqq i, j \leqq M$. Under the usual embedding $\iota\colon \boldsymbol{A}_K^{M^2} \to \boldsymbol{P}_K^{M^2}$, we have $\iota(x) = (1, a_{ij})$. A direct computation now gives the result.

COROLLARY 3.7. *Suppose $H = G \times A$ is the product of an affine group variety G and an abelian variety A, both defined over K. Then there exists a constant C'' such that, for all $x \in H(K)$, $\operatorname{size}(x^N) \leqq C''N^2 \operatorname{size}(x)$.*

As a corollary to Theorem 3.5, we obtain the theorem mentioned by Lang [5, p. 54].

THEOREM 3.8. *Let $Q(t)$ be a purely transcendental extension of Q of transcendence type $\leqq \tau$ for some integer $\tau \geqq 2$. Let K be the algebraic closure of $Q(t)$, A an abelian variety defined over K, $\varphi\colon \boldsymbol{C} \to A(\boldsymbol{C})$ a 1-parameter subgroup of A of algebraic dimension d, and Γ a subgroup of $\boldsymbol{C}$. Suppose Γ contains at least $2m+2$ elements $z_1, \ldots, z_{2m+2}$ which are linearly independent over Q, and such that $\varphi(\Gamma) \subset A(K)$. If $m \geqq d\tau$, then $\tau \geqq d$.*

REMARK. The methods used here do not extend to arbitrary commutative group varieties because Proposition 3.1 fails. For example if G is affine, the embedding of $G \hookrightarrow \boldsymbol{A}^{m^2} \hookrightarrow \boldsymbol{P}^{m^2}$ has $L = f^*\mathcal{O}_P(1)$, isomorphic to the structure sheaf of G. Since $\mathcal{O}_G^{\otimes N} \simeq \mathcal{O}_G$, Proposition 3.1 does not hold. On the other hand, if G is the commutative group variety of dimension 2 parametrized by $(1, \wp(t), \wp'(t), u - \zeta(t))$ where ζ is the Weierstrass zeta function [5, p. 43], then using the addition formulas for $\wp$ and ζ, one can verify directly that Proposition 2.1 holds in this case. Thus Theorem 3.8 holds not only for abelian varieties, but also for products $G \times A$ by Corollary 3.7 and for the group just mentioned.

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